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Comparison of Inverse Optimal and Tuning Functions Designs for Adaptive Missile Control

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I. Introduction

THE introduction of the adaptive backstepping approach [1] in the beginning of the 1990s led to a lot of interest in the flight control design community, mainly due to its strong convergence and stability properties and due to the fact that the method could be applied to a broad class of nonlinear systems. Adaptive backstepping techniques have been applied successfully in the last few years to solve various flight control problems, see, e.g., [2–4]. However, these adaptive control laws are all focused on achieving stability and convergence rather than performance or optimality, as is the case with most adaptive backstepping designs. The transient performance results achieved by the most widely used adaptive backstepping variant, the tuning functions approach [1], only provide performance estimates on the tracking error [5], not on the control effort. This would suggest combining or extending the Lyapunov-based control with some form of optimal control theory. However, the difficulty is that the direct optimal control problem for nonlinear systems requires the solving of a Hamilton–Jacobi–Bellman equation, which is not feasible in general. This obstacle motivated the development of *inverse* optimal nonlinear control design methods [6]. In the inverse approach, a positive definite Lyapunov function is given and the task is to determine whether a feedback control law minimizes some meaningful cost function, i.e., a cost that imposes a penalty on both the tracking errors and the control effort. The term inverse refers to the fact that the cost function is determined after the design of the stabilizing feedback control law, instead of being selected beforehand by the control designer. In [7], an inverse optimal adaptive backstepping control design for a general class of nonlinear systems has been derived.

In this paper, an inverse optimal approach based on theory presented in [7] is applied to the pitch autopilot design for a nonlinear model of an aerodynamically controlled generic surface-to-air missile with uncertainties in the aerodynamics. A second controller is designed by means of the well-known tuning functions adaptive backstepping technique, so that a comparison of both design methods can be made. The stability, convergence, and transient performance of both approaches are evaluated using numerical simulations with several levels of uncertainty in the aerodynamics and the control effectiveness of the missile model. The robustness of the control laws to sensor noise and (unmodeled) actuator dynamics, including rate, magnitude, and bandwidth limits, is also examined.

II. Missile Model Description

A second-order nonlinear model of a generic surface-to-air missile has been obtained from [3]. This model is well suited for developing and evaluating missile autopilots based on new control design methods: it is nonlinear, but not overly complex. The model consists of the longitudinal force and moment equations representative of a missile traveling at an altitude of approximately 6000 m, with aerodynamic coefficients represented as third-order polynomials in angle of attack α and Mach number M. The model is valid for the flight envelope $-10 < \alpha < 10$ deg and $1.8 < M < 2.6$. Based on the assumption that there are uncertainties in the aerodynamic force and moment terms, the uncertainty model for the longitudinal nonlinear missile model can be expressed as

$$\dot{x}_1 = x_2 + f_1(x_1)(1 + \Delta_{11}) + g_1(1 + \Delta_{12})u \tag{1}$$

$$\dot{x}_2 = f_2(x_1)(1 + \Delta_{21}) + g_2(1 + \Delta_{22})u \tag{2}$$

where Δ_{11}, Δ_{12}, Δ_{21}, and Δ_{22} represent bounded multiplicative unknown parameters and f_1, f_2, g_1, and g_2 are the aerodynamic forces and moments dependent on angle of attack and Mach number.

III. Missile Longitudinal Autopilot Design

The control objective considered here is angle-of-attack reference tracking. The short-period dynamics of the missile consisting of Eqs. (1) and (2) are sufficient for this missile flight control design problem [8]; the Mach number M is treated as a parameter available for measurement. Furthermore, the contribution of the fin deflection on the right-hand side of the force equation, Eq. (1), is neglected during the control design, because the adaptive backstepping method can only handle nonlinear systems of lower-triangular form [1], i.e., the assumption is made that the fin surface is a pure moment generator. This a valid assumption for this missile model. However, if the nonminimum phase behavior is more pronounced, a more elegant solution using the constrained adaptive backstepping approach of [9] can be used. Two types of adaptive backstepping controllers are designed for the uncertain missile model. In Sec. III.A, the well-known tuning functions approach [5] is applied, whereas the lesser-known inverse optimal adaptive backstepping approach [7] is used in Sec. III.B. The parameter update laws of both designs are robustified against parameter drift using dead zones and e-modification terms.

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A. Tuning Functions Design

First, the system Eqs. (1) and (2) are rewritten without the contribution of the fin deflection in the x_1 subsystem

$$\dot{x}_1 = x_2 + \varphi_1^T(x_1)\theta \tag{3}$$

$$\dot{x}_2 = \varphi_2^T(x_1)\theta + b_2 g_2 u \tag{4}$$

where $\theta = [1+\Delta_{11} \quad 1+\Delta_{21}]^T$ is an unknown constant parameter vector, $b_2 = 1+\Delta_{22}$ is an unknown constant parameter, and $\varphi_1 = [f_1 \quad 0]^T$ and $\varphi_2 = [0 \quad f_2]^T$ are regressor vectors. Because the control objective is to track a command signal $y_r(t)$ with the state x_1, and assuming the derivatives of y_r are known, the tracking errors are defined as

$$z_1 = x_1 - y_r \tag{5}$$

$$z_2 = x_2 - \alpha_1 - \dot{y}_r \tag{6}$$

where α_1 is the intermediate control law. Using the tuning functions design procedure, the following stabilizing functions are found

$$\alpha_1(x_1, y_r, \hat{\theta}) = -c_1 z_1 - \kappa_1|\varphi_1|^2 z_1 - \varphi_1^T\hat{\theta} \tag{7}$$

$$\begin{aligned}\alpha_2(x_1, x_2, y_r, \dot{y}_r, \hat{\theta}) = {} & -z_1 - c_2 z_2 - \kappa_2\left|\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right|^2 z_2 \\ & + \frac{\partial\alpha_1}{\partial x_1}x_2 - \left(\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right)^T\hat{\theta} + \frac{\partial\alpha_1}{\partial y_r}\dot{y}_r \\ & + \frac{\partial\alpha_1}{\partial\hat{\theta}}\Gamma\left[\left(\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right)z_2 + \varphi_1 z_1\right]\end{aligned} \tag{8}$$

where c_1, $c_2 > 0$ and κ_1, $\kappa_2 > 0$ are design constants, $\hat{\theta}$ is the parameter vector estimate, and $\Gamma = \Gamma^T > 0$ is the adaptation gain matrix. The nonlinear damping terms are included in the design to robustify the controller against estimation errors and to improve transient performance [5]. The update laws for the unknown parameter estimates are given by

$$\dot{\hat{\theta}} = \Gamma\left[\left(\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right)z_2 + \varphi_1 z_1\right] \tag{9}$$

$$\dot{\hat{\rho}} = -\gamma\,\mathrm{sgn}(b_2)(\alpha_2 + \ddot{y}_r)z_2 \tag{10}$$

where e-modification and dead-zone terms have been omitted for simplicity. Furthermore, $\hat{\rho}$ is the estimate of $\rho = 1/b_2$ and $\gamma > 0$ is the update gain. The estimate $\hat{\rho}$ is used to avoid singularity problems if the estimate of b_2 approaches the value of zero. Note that the assumption is made that the sign of b_2 is known. Alternatively, a parameter projection method can be used. Finally, the adaptive control law for u is given by

$$u = \frac{\hat{\rho}}{g_2}[\alpha_2(x_1, x_2, y_r, \dot{y}_r, \hat{\theta}) + \ddot{y}_r] \tag{11}$$

The control Lyapunov function is selected as

$$V = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} + \frac{|b_2|}{2\gamma}\tilde{\rho}^2 \tag{12}$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{\rho} = \rho - \hat{\rho}$ are the parameter estimation errors. The control law Eq. (11), together with the dynamic update laws Eqs. (9) and (10), renders the derivative of V equal to

$$\dot{V} = -c_1 z_1^2 - c_2 z_2^2 - \kappa_1|\varphi_1|^2 z_1^2 - \kappa_2\left|\varphi_2 - \frac{\partial\alpha_1}{\partial x_1}\varphi_1\right|^2 z_2^2 \le 0 \tag{13}$$

which proves that the equilibrium $(z, \tilde{\theta}, \tilde{\rho}) = 0$ is globally asymptotically stable. Furthermore, using Barbalat's lemma [1], it follows that

$$\lim_{t\to\infty} z(t) = 0$$

A block diagram of the tuning functions control law can be found in Fig. 1. To facilitate the tuning of the designed adaptive control law, the transient performance of the design is examined. In a way similar to the analysis in [5], and by assuming that the initial tracking error $z(0) = 0$, the following $\mathcal{L}_2$ and $\mathcal{L}_\infty$ performance bounds can be derived

$$\|z\|_2 \le \frac{1}{\sqrt{2c_0\gamma}}(|\tilde{\theta}(0)| + |b_2||\tilde{\rho}(0)|) \tag{14}$$

$$|z(t)| \le \min\left\{\frac{1}{2\sqrt{c_0\kappa_0}}, \frac{1}{\sqrt{\gamma}}\right\}(|\tilde{\theta}(0)| + |b_2||\tilde{\rho}(0)|) \tag{15}$$

where $c_0 = \min(c_1, c_2)$, $1/\kappa_0 = 1/\kappa_1 + 1/\kappa_2$, and, for simplicity, Γ has been taken equal to γI. It can be concluded that both performance bounds for the tuning functions scheme can be systematically reduced by increasing any of the design parameters c_0, κ_0, or γ.

B. Inverse Optimal Design

The tuning functions control design of the previous section is mainly focused on achieving stability. Several performance bounds can be derived for the tracking errors, the system states, and the estimated parameters, but these bounds do not contain any measurement of the required control effort. In this section, an inverse optimal adaptive backstepping control law is designed based on [7]. To facilitate the application of the inverse optimal method to the tracking control problem of the system Eqs. (3) and (4), the problem is first transformed into a regulation problem. For any *given* smooth reference function $y_r(t)$ there exist functions $\rho_1(t)$, $\rho_2(t, \theta)$, and

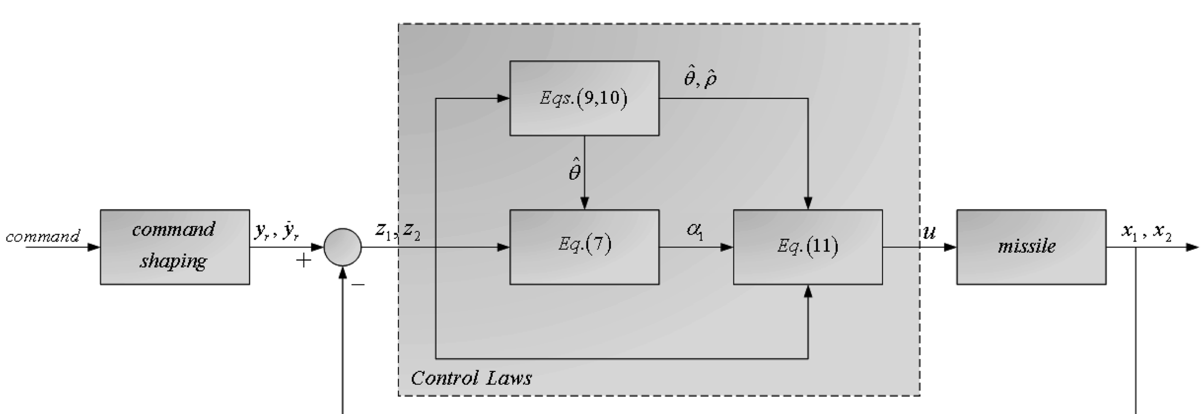

Fig. 1 Block diagram of tuning functions control law.

$\alpha_r(t, \theta)$, such that

$$\dot{\rho}_1 = \rho_2 + \varphi_1^T(\rho_1)\theta \qquad \dot{\rho}_2 = \alpha_r(t, \theta) + \varphi_2^T(\rho_1)\theta \\ y_r(t) = \rho_1(t) \tag{16}$$

Since $\partial y_r/\partial\theta = 0$, it follows that $\partial\rho_1/\partial\theta = 0$ for all $t \geq 0$ and for all $\theta \in \mathbb{R}^2$, θ can be replaced by its estimate $\hat{\theta}$. Consider the signal $x_r(t) = \rho(t, \hat{\theta}(t))$, which is governed by

$$\dot{x}_{r1} = x_{r2} + \varphi_{r1}^T(x_{r1})\hat{\theta} \qquad \dot{x}_{r2} = \alpha_r(t, \hat{\theta}) + \varphi_{r2}^T(x_{r1})\hat{\theta} + \tfrac{\partial\rho_2}{\partial\hat{\theta}}\dot{\hat{\theta}} \\ y_r(t) = x_{r1}(t) \tag{17}$$

Basically, we have constructed a reference model using the reference signal $y_r(t)$. Defining the tracking error $e = x - x_r$, the system can be rewritten as

$$\dot{e}_1 = e_2 + \tilde{\varphi}_1^T(e_1, \hat{\theta})\theta + \varphi_{r1}^T(x_{r1}, \hat{\theta})\tilde{\theta} \tag{18}$$

$$\dot{e}_2 = \tilde{u} + \tilde{\varphi}_2^T(e_1, \hat{\theta})\theta + \varphi_{r2}^T(x_{r1}, \hat{\theta})\tilde{\theta} - \frac{\partial\rho_2}{\partial\hat{\theta}}\dot{\hat{\theta}} \tag{19}$$

where $\tilde{u} = b_2 g_2 u - \alpha_r(t, \hat{\theta})$ and $\tilde{\varphi}_i = \varphi_i(x_1, \ldots, x_i) - \varphi_{ri}(x_{r1}, \ldots, x_{ri})$, $i = 1, 2$. Now, the inverse optimal tracking problem has been transformed into an inverse optimal regulation problem. Again, new states are defined as

$$z_1 = e_1 \qquad z_2 = e_2 - \tilde{\alpha}_1 \tag{20}$$

where $\tilde{\alpha}_1$ is the intermediate control law. Note that z_2 is defined the same as in Eq. (12) of Sec. III.A. The first task is now to stabilize the system Eqs. (18) and (19), after which the second task is to find a meaningful cost function and proof of optimality. As in the previous section, the tuning functions method is used to determine the intermediate control

$$\tilde{\alpha}_1(e_1, \hat{\theta}) = -c_1 z_1 - \tilde{\varphi}_1^T\hat{\theta}, \qquad c_1 > 0 \tag{21}$$

which is very similar to Eq. (7) of the tuning functions design minus the nonlinear damping term. A control Lyapunov function V_2 is now selected as

$$V_2 = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2 + \frac{1}{2}\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{22}$$

After applying the virtual control law Eq. (21), the derivative of V_2 becomes

$$\dot{V}_2 = -c_1 z_1^2 + z_2\left[z_1 + \tilde{u} - \frac{\partial\tilde{\alpha}_1}{\partial e_1}e_2 + \left(\tilde{\varphi}_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\tilde{\varphi}_1\right)^T\hat{\theta} - \left(\frac{\partial\tilde{\alpha}_1}{\partial\hat{\theta}} + \frac{\partial\rho_2}{\partial\hat{\theta}}\right)\dot{\hat{\theta}}\right] + \tilde{\theta}^T\left[\varphi_1 z_1 + \left(\varphi_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\varphi_1\right)z_2 - \Gamma^{-1}\dot{\hat{\theta}}\right] \tag{23}$$

To eliminate $\tilde{\theta}$ from $\dot{V}_2$, an update law similar to Eq. (9) is chosen. This reduces the expression for $\dot{V}_2$ to

$$\dot{V}_2 = -c_1 z_1^2 + z_2\left\{z_1 + \tilde{u} - \frac{\partial\tilde{\alpha}_1}{\partial e_1}e_2 + \left(\tilde{\varphi}_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\tilde{\varphi}_1\right)^T\hat{\theta} - \left(\frac{\partial\tilde{\alpha}_1}{\partial\hat{\theta}} + \frac{\partial\rho_2}{\partial\hat{\theta}}\right)\Gamma\left[\left(\varphi_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\varphi_1\right)z_2 + \varphi_1 z_1\right]\right\} \tag{24}$$

Now the control $\tilde{u}$ can be designed to render $\dot{V}_2$ negative definite. However, applying the tuning functions method would result in the cancellation of all indefinite terms, and the resulting control law cannot be shown to be inverse optimal with respect to a meaningful cost function. In [7], a control of the form

$$\tilde{u} = \tilde{\alpha}_2(e, \hat{\theta}) = -r^{-1}(e, \hat{\theta})\frac{\partial V}{\partial e}g, \qquad r(e, \hat{\theta}) > 0 \quad \forall\ t, e, \hat{\theta} \tag{25}$$

is suggested, based on theory in Sec. 3.5 of [10]. For this control problem, Eq. (25) simplifies to

$$\tilde{u} = \tilde{\alpha}_2(e, \hat{\theta}) = -r^{-1}(e, \hat{\theta})z_2 \tag{26}$$

i.e., z_2 has to be a factor of the control. To get rid of the indefinite terms without canceling them, nonlinear damping is applied. Because the terms $(-\partial\tilde{\alpha}_1/\partial e_1)e_2$ and $[\tilde{\varphi}_2 - (\partial\tilde{\alpha}_1/\partial e_1)\tilde{\varphi}_1]^T\hat{\theta}$ vanish at $z_1 = z_2 = 0$, there exist smooth functions ϕ_1 and ϕ_2 such that

$$-\frac{\partial\tilde{\alpha}_1}{\partial e_1}e_2 + \left(\tilde{\varphi}_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\tilde{\varphi}_1\right)^T\hat{\theta} = \phi_1 z_1 + \phi_2 z_2 \tag{27}$$

Thus, Eq. (23) becomes

$$\dot{V}_2 = -c_1 z_1^2 + z_2\tilde{u} + z_1\Phi_1 z_2 + z_2\Phi_2 z_2 \tag{28}$$

where

$$\Phi_1 = 1 - \left(\frac{\partial\tilde{\alpha}_1}{\partial\hat{\theta}} + \frac{\partial\rho_2}{\partial\hat{\theta}}\right)\Gamma\varphi_1 + \phi_1 \tag{29}$$

$$\Phi_2 = -\left(\frac{\partial\tilde{\alpha}_1}{\partial\hat{\theta}} + \frac{\partial\rho_2}{\partial\hat{\theta}}\right)\Gamma\left(\varphi_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\varphi_1\right) + \phi_2 \tag{30}$$

A control law of the form of Eq. (26) with

$$r(e, \hat{\theta}) = \left(c_2 + \frac{\Phi_1^2}{2c_1} + \frac{\Phi_2^2}{2c_2}\right)^{-1} > 0, \qquad c_2 > 0, \quad \forall\ t, e, \hat{\theta} \tag{31}$$

containing the nonlinear damping terms, results in

$$\dot{V}_2 = -\frac{1}{2}c_1 z_1^2 - \frac{1}{2}c_2 z_2^2 - \frac{c_1}{2}\left(z_1 - \frac{\Phi_1}{c_1}z_2\right)^2 - \frac{c_2}{2}\left(z_2 - \frac{\Phi_2}{c_2}z_2\right)^2 < 0, \qquad z_1, z_2 \neq 0 \tag{32}$$

It can be concluded that the closed-loop system is stabilized, since $\dot{V}_2$ is negative definite. However, up until now it was neglected that the actual control $u = (\rho/g_2)(\tilde{u} + \alpha_r)$ and that the parameter $\rho = 1/b_2$ is unknown. Using the control Lyapunov function V as defined in Eq. (12) of Sec. III.A, an additional parameter update law for $\hat{\rho}$ is chosen as

$$\dot{\hat{\rho}} = -\gamma\,\mathrm{sgn}(b_2)(-r^{-1}z_2 + \alpha_r)z_2 \tag{33}$$

to complete the control design. Note that Eq. (33) is not the same as the update law for $\hat{\rho}$ used in the tuning functions design. Now, the second task is to show that the dynamic feedback control law is inverse optimal with respect to a meaningful cost function. Based on the theory of [7], a cost function of the form

$$J = \beta\lim_{t\to\infty}|\theta - \hat{\theta}(t)|_{\Gamma^{-1}}^2 + \beta|b_2|\lim_{t\to\infty}|\rho - \hat{\rho}(t)|_{\gamma^{-1}}^2 + \int_0^\infty[l(e, \hat{\theta}) + r(e, \hat{\theta})\tilde{u}^2]\,\mathrm{d}t, \quad \forall\ \theta \in \mathbb{R}^2 \tag{34}$$

is selected, where

$$l(e, \hat{\theta}) = -2\beta\dot{V} + \beta(\beta - 2)r^{-1}z_2^2 \tag{35}$$

Since $\beta \geq 2$, $r(e, \hat{\theta}) > 0$, and $\dot{V}$ is negative definite, it is clear that $l(e, \hat{\theta})$ is positive definite. Therefore, J defined in Eq. (34) is a "meaningful" cost function which puts a penalty on both z and $\tilde{u}$, as well as on the terminal values of $|\tilde{\theta}|$ and $|\tilde{\rho}|$. An integral penalty on the parameter estimation errors is not included, because the theory behind the adaptive backstepping approach does not guarantee parameter estimation error convergence to zero [1]. There remains to demonstrate that the dynamic feedback control law

$$\tilde{u} = \tilde{\alpha}_2^*(e, \hat{\theta}) = \beta\tilde{\alpha}_2(e, \hat{\theta}) = -\beta\left(c_2 + \frac{\Phi_1^2}{2c_1} + \frac{\Phi_2^2}{2c_2}\right)^{-1} z_2$$
$$\dot{\hat{\theta}} = \Gamma\left[\left(\varphi_2 - \frac{\partial\tilde{\alpha}_1}{\partial e_1}\varphi_1\right)z_2 + \varphi_1 z_1\right] \tag{36}$$
$$\dot{\hat{\rho}} = -\gamma[\mathrm{sgn}(b_2)(-rz_2 + \alpha_r)z_2]$$

does not only stabilize the error system Eqs. (18) and (19) with respect to the control Lyapunov function, Eq. (12), but is also optimal with respect to the cost function Eq. (34). The proof requires the definition of the static part of the control Lyapunov function Eq. (12)

$$V_s = \frac{1}{2}z_1^2 + \frac{1}{2}z_2^2$$

Substituting $l(e, \hat{\theta})$ and

$$v = \tilde{u} - \tilde{\alpha}_2^* = \tilde{u} + \beta r^{-1}(e, \hat{\theta})z_2 \tag{37}$$

into J, together with Eq. (28), gives

$$J = \beta|\tilde{\theta}(0)|^2_{\Gamma^{-1}} + \beta|b_2||\tilde{\rho}(0)|^2_{\gamma^{-1}} + 2\beta V_s(z(0), \hat{\theta}(0))$$
$$- 2\beta\lim_{t\to\infty} V_s(z(t), \hat{\theta}(t)) + \int_0^\infty rv^2\,\mathrm{d}t \tag{38}$$

It was already shown that the control law Eq. (26), together with the update laws for $\hat{\theta}$ and $\hat{\rho}$, stabilizes the closed-loop system, which means

$$\lim_{t\to\infty} z(t) = 0$$

and

$$\lim_{t\to\infty} V_s(z(t), \hat{\theta}(t)) = 0$$

Therefore, the minimum of Eq. (38) is reached only if $v = 0$, and thus the control law $\tilde{u} = \tilde{\alpha}_2^*$ minimizes the cost function Eq. (34). Summarizing, if $\beta = 2$, the dynamic control law Eq. (36) is optimal with respect to the cost function

$$J = 2\lim_{t\to\infty}|\tilde{\theta}|^2_{\Gamma^{-1}} + 2|b_2|\lim_{t\to\infty}|\tilde{\rho}|^2_{\gamma^{-1}} + 2\int_0^\infty\left[c_1 z_1^2 + c_2 z_2^2\right.$$
$$+ c_1\left(z_1 - \frac{\Phi_1}{c_1}z_2\right)^2 + c_2\left(z_2 - \frac{\Phi_2}{c_2}z_2\right)^2$$
$$\left. + \frac{\tilde{u}^2}{2[c_2 + (\Phi_1^2/2c_1) + (\Phi_2^2/2c_2)]}\right]\mathrm{d}t \tag{39}$$

with the value function

$$J^* = 2|\tilde{\theta}|^2_{\Gamma^{-1}} + 2|b_2||\tilde{\rho}|^2_{\gamma^{-1}} + 2|z|^2 \tag{40}$$

According to Theorem 3 in [7], the following $\mathcal{L}_2$ performance bound holds for the system Eqs. (18) and (19) and the dynamic control law Eq. (36), when $z(0) = 0$:

$$\int_0^\infty\left[c_1 z_1^2 + c_2 z_2^2 + \frac{\tilde{u}^2}{2[c_2 + (\Phi_1^2/2c_1) + (\Phi_2^2/2c_2)]}\right]\mathrm{d}t$$
$$\leq |\tilde{\theta}(0)|^2_{\Gamma^{-1}} + |b_2||\tilde{\rho}(0)|^2_{\gamma^{-1}} \tag{41}$$

The control effort is included in this performance bound, contrary to the bounds derived for the tuning functions design Eqs. (14) and (15) in Sec. III.A. This concludes the inverse optimal adaptive design. A block diagram for the inverse optimal adaptive design is omitted, because it is almost identical to the diagram of Fig. 1.

IV. Simulation Results

The control laws developed in the previous section are now applied to the generic nonlinear missile model introduced in Sec. IV. Numerical simulations performed in MATLAB/Simulink are used to compare both designs.

A. Gain Selection Process

The control laws can be tuned in a trial-and-error procedure with the help of the performance bounds Eqs. (14), (15), and (41). The reference signal and its derivative are generated by a second-order linear command filter. The controller parameters that were eventually chosen can be found in Table 1. From the values in this table, it appears that the control laws are almost tuned the same. However, examining the performance bounds Eqs. (14), (15), and (41) reveals that this not the case. The control law of the inverse optimal design contains the large nonlinear damping terms $\Phi_1^2/2c_1$ and $\Phi_2^2/2c_2$; especially the first term can grow quite large as it contains the derivatives of the virtual control law. Because of the fast nonlinear growth resulting from these terms, the control law is numerically very sensitive. It is difficult to reduce $\Phi_1^2/2c_1$, since Φ_1 is also dependent on c_1. For other control applications where the derivative of the intermediate control law is much smaller, e.g., attitude control, the design approach may be beneficial, because the nonlinear growth will be more restricted.

The gain selection process for the tuning functions controller is much more straightforward. For this control design method, gain selection vs numerical stability was already studied in detail in [11]. As can be seen from Eqs. (14) and (15), the convergence rate of the tuning functions design can be increased by enlarging the update gains, control gains, or the damping gains. Weak damping terms make the control law robust to parameter estimation errors [1]. The parameters in Table 1 result in very different time responses of both closed-loop systems. The response with the inverse optimal control law is much more aggressive, but reducing the convergence rate of the inverse optimal control law is not possible due to the large nonlinear damping terms. The tuning functions controller can be tuned to respond as aggressively as the inverse optimal design, but it will also become very numerically sensitive. To better illustrate this shortcoming of the latter design, it was opted not to tune the tuning functions as aggressively. Of course, it would be more "fair" to tune the controllers the same in a comparison. However, with the tuning functions design, more tuning possibilities exist. This is not the case with the inverse optimal design and it should also be taken into account in the comparison. The gains of the tuning functions design have been selected mainly with performance in mind, whereas the tuning process of the inverse optimal design was hampered by the fact that numerical stability has to be maintained. Note, however, that the dynamic part of the controllers is tuned the same.

B. Simulations with Aerodynamic Uncertainties

In this section, the ideal case is considered in which the actuator dynamics and sensor noise are not included in the simulation; only the aerodynamic uncertainties which are taken into account by the adaptive designs are considered. The control law parameters are given in Table 1. Numerical simulations were performed for two uncertainty cases, not counting the nominal case. For both cases, large modeling errors in the aerodynamics were considered. For uncertainty case A, $\Delta_{11} = 0.4$, $\Delta_{12} = -0.4$, $\Delta_{21} = -0.4$, and

Table 1 Control law parameters

Tuning functions	Inverse optimal
$c_1 = 8$	$c_1 = 10$
$c_2 = 10$	$c_2 = 10$
$\Gamma = \begin{bmatrix} 100 & 0 \\ 0 & 0.01 \end{bmatrix}$	$\Gamma = \begin{bmatrix} 100 & 0 \\ 0 & 0.01 \end{bmatrix}$
$\gamma = 0.01$	$\gamma = 0.01$
$\kappa_1 = \kappa_2 = 0.04$	

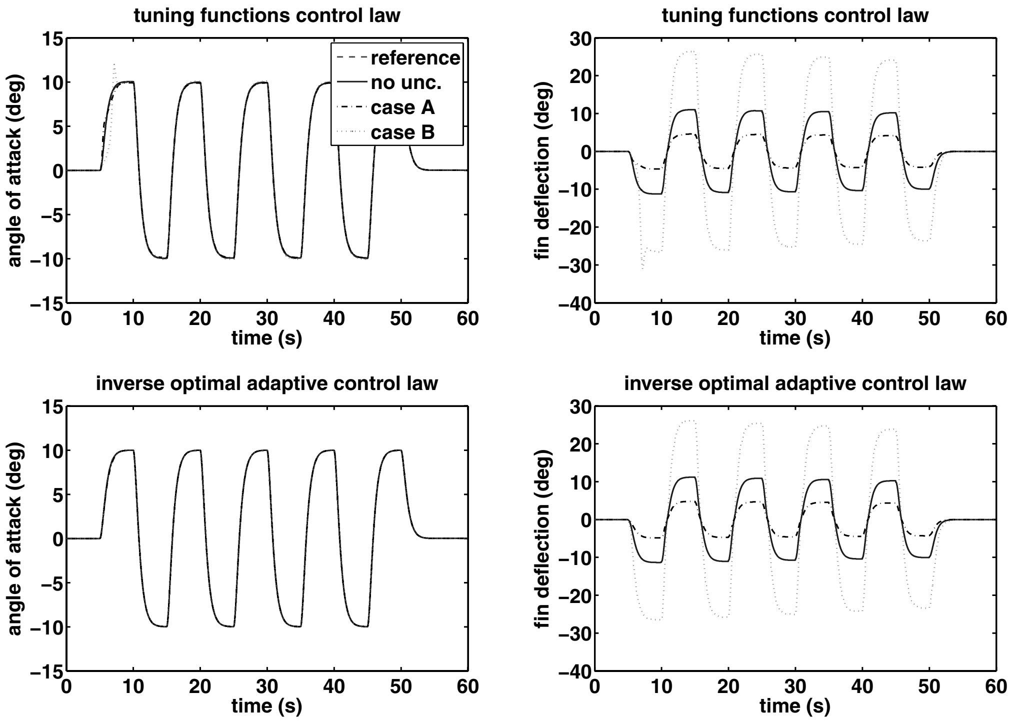

Fig. 2 Angle-of-attack α response and control effort u for both controllers.

$\Delta_{22} = 0.4$, whereas for case B, all uncertainties have the same size but a different sign. Numerical results of one of the tracking maneuvers with the tuning functions controller can be found in the upper plots of Fig. 2. The Mach number during the simulation rises from 1.8 to 2.6. In the figure, the dashed line corresponds to the desired angle of attack, the solid line represents the nominal case, and the two other lines indicate the uncertainty cases A and B. It can be seen that this adaptive control law is able to identify the uncertainties and track the reference signal. However, at the first step, the transient performance leaves something to be desired, especially for case B. This peak becomes higher if the weak nonlinear damping terms are removed. Plots of the parameter estimates are not includes due to space limitations. However, note that parameter convergence to zero is not guaranteed and also not necessary for an adaptive backstepping controller to achieve stability. The performance of the inverse optimal adaptive control law of Sec. III.B is examined. Again the same maneuver has been simulated, and the results can be found in the lower plots of Fig. 2. Notice that this adaptive controller is also able to track the reference signal very well. As was expected, the transient performance is much better at the first step command, because this control law is tuned much more aggressively.

C. Simulations with Actuator Dynamics

In the ideal uncertainty case, both adaptive control laws are able to accurately track the command signal, but the transient performance of the inverse optimal control law is much better. However, in the presence of unmodeled actuator dynamics with rate, magnitude, and bandwidth limits, the performance of the adaptive control laws may degrade, especially for the inverse optimal design. To investigate the robustness properties of the control laws, these effects are now considered in the simulations. The missile's tail fin actuator is modeled as a second-order system with a rate limit of 100 deg/s and a deflection limit of ± 45 deg. Two actuator bandwidths are considered: 50 and 10 Hz. The controller parameters remain the same as in the ideal case of the previous section and the missile still flies at a varying airspeed. The missile model with the uncertainties of case A has been taken. Numerical results for the missile with tuning functions control law are shown in the top plots of Fig. 3. Notice the initial mediocre performance due to the parameter adaptation. The tracking performance of the adaptive controller does not visibly degrade when the actuator with a bandwidth of 50 Hz is used, but fluctuations occur in the state and control when the bandwidth is taken at only 10 Hz. Other simulations show that the performance of the control law remains acceptable up until bandwidths as low as 15 Hz. The results of the same scenario using the inverse optimal adaptive control law can be found in the lower plots of Fig. 3. This control law can also deal with 50 Hz bandwidth, but the response becomes very unstable if the 10 Hz actuator is used. A study demonstrated that the response of the missile with the inverse optimal control law already starts to fluctuate at a bandwidth of 40 Hz, which also happens with the tuning functions controller when tuned as aggressively.

D. Simulations with Sensor Noise

In these final simulations, sensor noise is added to the system. The noise is assumed to be zero mean uncorrelated white noise with a state covariance matrix

$$E(XX^T) = \mathrm{diag}\left[\left(0.5\frac{\pi}{180}\right)^2 \quad 10^{-4}\right]$$

The controller parameters remain unchanged and, again, uncertainty case A is considered. The upper plots of Fig. 4 show the results of a simulation with the tuning functions control law. It can be seen that the control law still gives satisfying tracking performance despite the measurement noise on the system state. Some high-gain behavior is also visible, but reducing the weak damping terms can reduce the peaks. The results of the simulation for the missile with the inverse optimal control law can be found in the bottom plots of Fig. 4. The top plot demonstrates that the tracking performance is hardly affected by the noise, but the bottom plot shows that the control signal is now chattering heavily. Because of the heavy use of nonlinear damping terms, the inverse optimal control law tries to completely dampen out the noise, i.e., high-gain control. Tightening the bound on the control as far as possible, and thus reducing the bound on the tracking errors, reduces the chattering in the control only marginally. Adding the actuator with a bandwidth of 50 Hz to the simulation does not change the simulation results for the tuning functions control law, but the numerical simulation with inverse optimal control law crashes immediately.

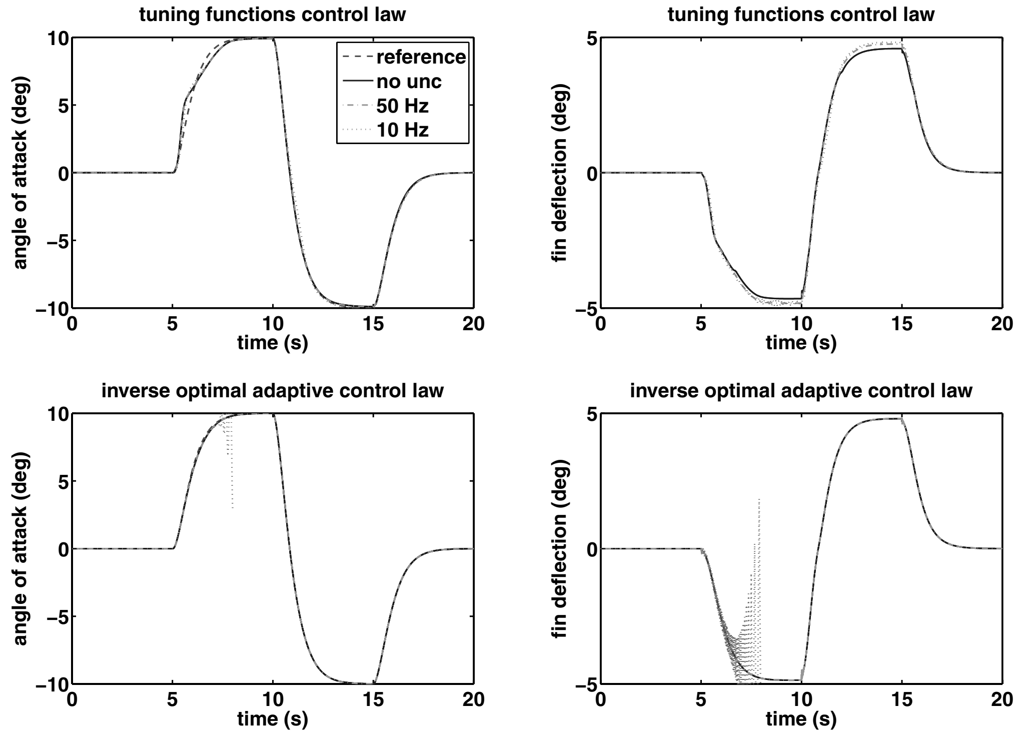

Fig. 3 Angle-of-attack α response and control effort u for simulations with actuator dynamics for both controllers.

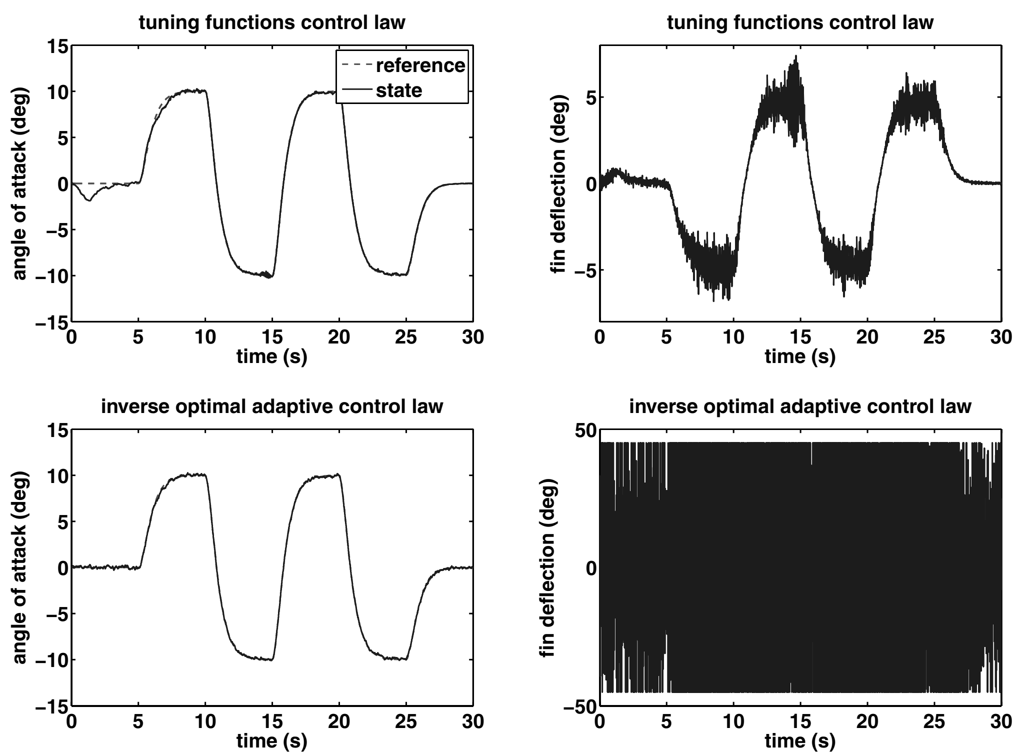

Fig. 4 Angle-of-attack α response and control effort u for simulations with sensor noise for both controllers.

V. Conclusions

In this study, two adaptive backstepping control design techniques have been applied to the design of a pitch autopilot for a nonlinear generic surface-to-air missile model with large uncertainties in the aerodynamic and control surface effectiveness parameters. In numerical simulations using the ideal missile model with uncertainties, the inverse optimal adaptive design outperformed the tuning functions design in terms of transient performance. However, the robustness of the inverse optimal control law is somewhat disappointing, as is shown in numerical simulations where unmodeled actuator dynamics and large sensor noise have been added. The tuning functions adaptive backstepping design can be made quite robust to these effects and thus is more suited as an autopilot for a real missile.

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